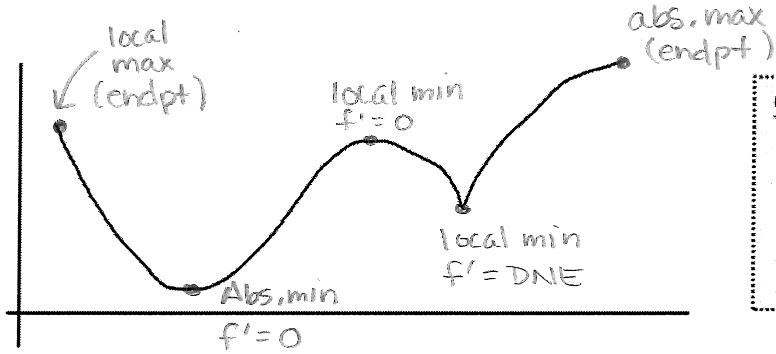
Chapter 4 notes Name Key (2014)

4.1: Extreme Values of Functions

EXAMPLE: Where do the extreme values occur?



CRITICAL POINT DEFINITION:

Any point in interior where $f'=0$ or f' DNE

ABSOLUTE (global): f is a function with domain D

Max – no greater y value anywhere

Min – no lessor y value anywhere

LOCAL (relative): c is an interior point of domain D of f

Max – no greater y value in "neighborhood"

Min – no lessor y value in "neighborhood"

x values y values

EXAMPLE: Where do the extreme values occur? What are the extreme values?

1If D = (0, 2]

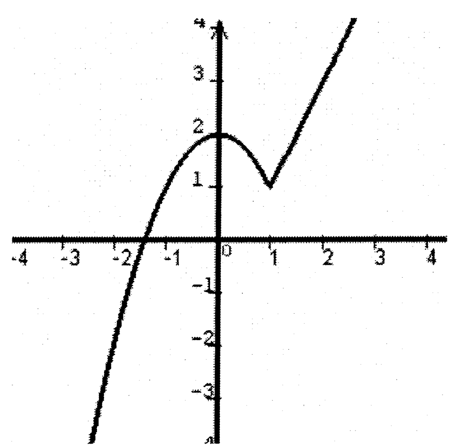
where?	what?
x=1 (f'=DNE)	y=2 (local min)
x=2 (endpt)	y=3 (abs. max)

2If D = (-∞,∞)

where?	what?
x=0 (f'=0)	y=1.8 (local max)
x=1 (f'=DNE)	y=2 (local min)

3If D = [-2, 2]

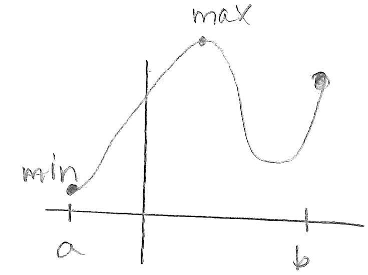
Where?	What?
x=-2 (endpt)	y=-2 (abs. min)
x=0 (f'=0)	y=1.8 (local max)
x=1 (f'=DNE)	y=2 (local min)
x=2 (endpt)	y=3 (abs. max)



EXTREME VALUE THEOREM:

> **If f is continuous on [a, b],**
> then f has both a max. and min. on the interval.

$\underbrace{\hspace{4em}}_{\text{absolute}}$



EXAMPLE: Use analytical methods to find extreme values.

1. $f(x) = x^3 + x^2 - 8x + 5 \quad -3 \le x \le 4$

$f' = 3x^2 + 2x - 8$

$f' = 0$	$f' = \text{DNE}$	endpts
$3x^2 + 2x - 8 = 0$	none	$x = -3, 4$
$(3x-4)(x+2) = 0$		
$x = 4/3, -2$		

	-3 to -2	-2 to $4/3$	$4/3$ to 4
$(3x-4)$	$-$	$-$	$+$
$(x+2)$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	min (-3)	max (-2)	min ($4/3$), max (4)

extrema values

to see if absolute min or max

"candidate chart"

x	$f(x)$
-3	11 (local) max: left end, $f' > 0$
-2	17 (local) max: f' + to $-$
$4/3$	$-41/27$ (abs) min: f' $-$ to $+$
4	53 (abs) max: right end, $f' > 0$

2. $f(x) = \sqrt{3 + 2x - x^2} = \sqrt{(3-x)(1+x)}$ Domain: $[-1, 3]$

$f' = \frac{1}{2}(3 + 2x - x^2)^{-1/2}(2 - 2x)$

$f' = \dfrac{2 - 2x}{2\sqrt{3 + 2x - x^2}}$

$f' = \dfrac{2(1-x)}{2\sqrt{3 + 2x - x^2}}$

$f' = \dfrac{1-x}{\sqrt{3 + 2x - x^2}}$

$f' = 0$	$f' = \text{DNE}$	endpts
$1 - x = 0$	$\sqrt{3-x} \cdot \sqrt{1+x} = 0$	$x = -1, 3$
$x = 1$	$x = 3, -1$	

	-1 to 1	1 to 3
$(1-x)$	$+$	$-$
$\sqrt{3-x}$	$+$	$+$
$\sqrt{1+x}$	$+$	$+$
f'	$+$	$-$
f	min (-1), max (1)	min (3)

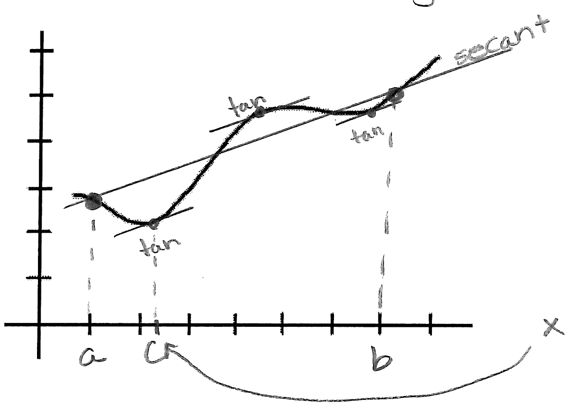
x	y
-1	0 (abs) min, left end $f' > 0$
1	2 (abs) max, f' + to $-$
3	0 (abs.) min, right end $f' < 0$

4.2: Extreme Values of Functions

Mean Value Theorem for Derivatives

If y = f(x) is continuous at every point of the closed interval [a, b] and differentiable at every point of its interior (a, b), then
there is at least one pt. c in (a,b) where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

slope of tangent — slope of secant



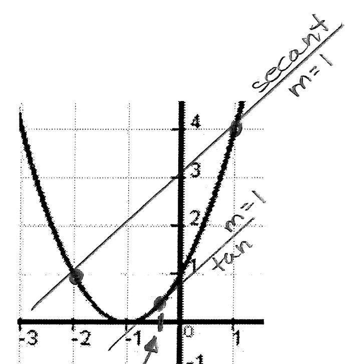
x value where slope of tangent = slope of secant (parallel to)

Example: $f(x) = x^2 + 2x + 1 \quad [-2, 1]$

a) Check to see that f(x) satisfies the MVT hypotheses.

b) Find c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

a) continuous on [-2,1]? Yes
differentiable on (-2,1) Yes



b) slope of tan

$f' = 2x + 2$

slope of secant

$\dfrac{f(1)-f(-2)}{1-(-2)} = \dfrac{4-1}{3} = 1$

$2x+2 = 1$
$2x = -1 \longrightarrow \boxed{x = -\tfrac{1}{2}}$

this is where slope of tan is parallel to slope of secant

Corollary:

f is continuous on [a,b] and differentiable on (a,b)

- f increases on (a,b) if $f' > 0$ at each pt. on (a,b)
- f decreases on (a,b) if $f' < 0$ at each pt. on (a,b)

Example: Where is $f(x) = x^2 + 2x + 1$ increasing and decreasing?

1) find where $f' = 0$ or $f' =$ DNE

$f' = 2x + 2$

$0 = 2x + 2$

$x = -1$

	-1	
$f' = 2x+2$	−	+
f	↘ dec	↗ inc

2) make sign chart of f'

f(x):
dec $(-\infty, -1)$
inc $(-1, \infty)$

EXAMPLE: Given $f(x) = x^{8/3} - x^{2/3}$

- find extreme values
- find intervals where increasing
- find intervals where decreasing

$$f' = \frac{8}{3}x^{5/3} - \frac{2}{3}x^{-1/3} = \frac{8}{3}x^{5/3} - \frac{2}{3x^{1/3}} = \frac{8x^{5/3}}{3}\cdot\frac{x^{1/3}}{x^{1/3}} - \frac{2}{3x^{1/3}} = \frac{8x^2-2}{3x^{1/3}}$$

$f'=0$	$f'=$ DNE	endpts
$8x^2-2=0$	$3x^{1/3}=0$	none
$x^2 = 1/4$	$x=0$	D: $(-\infty,\infty)$
$x = \pm 1/2$		

sign chart:

		$-1/2$		0		$1/2$	
$8x^2-2$	+		−		−		+
$3x^{1/3}$	−		−		+		+
f'	−		+		−		+
f	↘	min	↗	max	↘	min	↗

candidate chart:

extrema values

x	$f(x)$	
$-1/2$	$-.472$	(abs) min b/c f' − to +
0	0	(local) max b/c f' + to −
$1/2$	$-.472$	(abs) min b/c f' − to +

inc: $(-1/2, 0), (1/2, \infty)$
dec: $(-\infty, -1/2), (0, 1/2)$

COROLLARY:

Functions with the same derivative differ by a constant.

$f(x) = 4x^3 - 11x + 3 \qquad f'(x) = 12x^2 - 11$

$g(x) = 4x^3 - 11x - 7 \qquad g'(x) = 12x^2 - 11$

$h(x) = 4x^3 - 11x + \pi \qquad h'(x) = 12x^2 - 11$

EXAMPLE: Find the function thru (2, 1) with the given derivative.
$f'(x) = -x^{-2}, \qquad x > 0$

↑ use this to find c

$(2,1) = (x, f(x))$

$f(x) = -\frac{x^{-1}}{-1} + C$

$f(x) = \frac{1}{x} + C$

$1 = \frac{1}{2} + C$

$-\frac{1}{2} = C$

$$\boxed{f(x) = \frac{1}{x} - \frac{1}{2}}$$

"Antidifferentiation" finding the antiderivative

Definition: Antiderivative

A function F(x) is an antiderivative of a function f(x) if
$F'(x) = f(x)$ for all x

↑ derivative of antiderivative

Antiderivative Power Rule
If $f'(x) = x^n$
$$f(x) = \frac{x^{n+1}}{n+1} + C$$

EXAMPLE: $f'(x) = 3x^2$, $x > 0$ and $P(1,7)$
Find the function thru P with given the derivative.

$f(x) = \frac{3x^3}{3} + C$

$f(x) = x^3 + C$

$7 = 1^3 + C$

$6 = C$

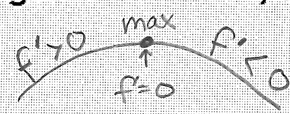
$$\boxed{f(x) = x^3 + 6}$$

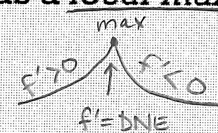
4.3 part I: First Derivative Test

FIRST DERIVATIVE TEST FOR LOCAL EXTREMA

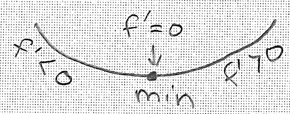
AT A CRITICAL POINT C: $f'=0$ or $f'=DNE$

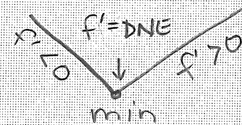
① If f' changes from + to -, then f has a *local max*. at c.



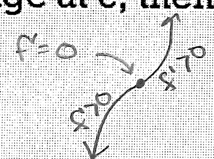


② If f' changes from - to +, then f has a *local min.* at c.



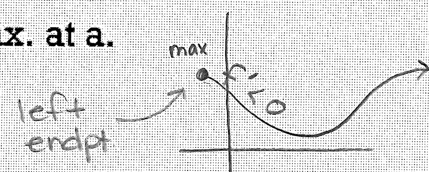


③ If f' has no sign change at c, then f has *no* local extrema at c. (might be inflection pt)

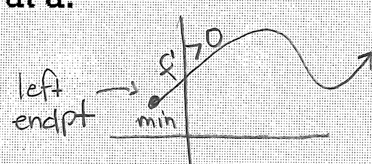


AT LEFT ENDPOINT A:

If f' < 0, then f has a local max. at a.

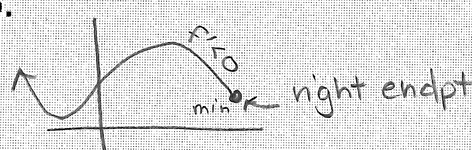


If f' > 0, then f has a local min. at a.

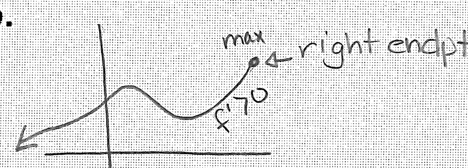


AT RIGHT ENDPOINT B:

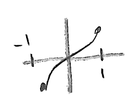
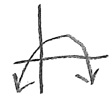
If f' < 0, then f has a local min. at b.



If f' > 0, then f has a local max. at b.



EXAMPLE: What are the endpoints of the functions? → sometimes endpts given, other-times, check domain

$f(x) = \sqrt{4-x^2}$ 	$g(x) = -5x^6 + 3x^2 - 1$	$h(x) = \sin^{-1}x$ 
D: $[-2, 2]$	D: $(-\infty, \infty)$ 	D: $[-1, 1]$
endpts: $x = -2, 2$	no endpts	endpts $x = -1, 1$

EXAMPLE: Given $y = 4x^3 + 21x^2 + 36x - 20$ on $[-4, 3]$

- Use the 1st Derivative Test to find extrema
- State intervals where increasing and decreasing

$f' = 12x^2 + 42x + 36 = 6(2x^2 + 7x + 6)$

$\underline{f' = 0}$

$0 = 6(2x^2 + 7x + 6)$

$0 = 6(2x + 3)(x + 2)$

$x = -3/2, -2$

$\underline{f' = DNE}$

none

$\underline{\text{endpts}}$

$x = -4, 3$

	-4 to -2	-2 to $-3/2$	$-3/2$ to 3
$6(2x+3)$	$-$	$-$	$+$
$(x+2)$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f			

min (-4), max (-2), min ($-3/2$), max (3)

$\boxed{\begin{array}{l} \text{inc: } [-4, -2), (-3/2, 3] \\ \text{dec: } (-2, -3/2) \end{array}}$

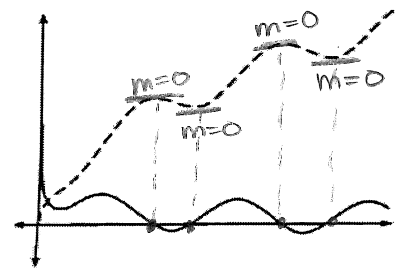
extrema values

x	$f(x)$
-4	-596 (abs) min b/c L end, $f' > 0$
-2	-40 (local) max b/c f' + to −
$-3/2$	-40.25 (local) min b/c f' − to +
3	601 (abs) max b/c R end, $f' > 0$

EXAMPLE: Which is g(x) and which is g'(x)?

dotted   solid

where g(x) [dotted] has slope of zero, g'(x) [solid] has value of zero



TEST: CONCAVITY

The graph of a twice differentiable function y = f(x) is

- concave up when $y'' > 0$
- concave down when $y'' <$

DEFINITION: POINT OF INFLECTION

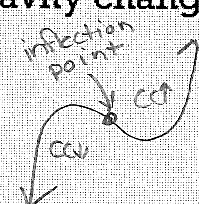
A point where there is a tangent line and the concavity changes.

either

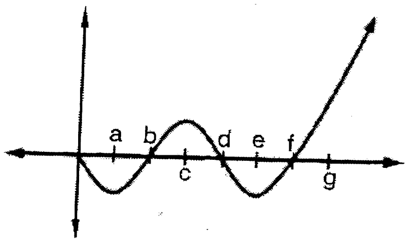
$f'' = 0$ and changes sign

or

$f'' = DNE$ and changes sign



Example: Below is a graph of h''(x)



a) On what intervals is h(x) concave up?

Where $h'' > 0$ $(b,d), (f,\infty)$

b) On what intervals is h'(x) concave down?

Where $h''' < 0$ (slope of h'') $(0,a), (c,e)$

c) What are the inflection points of h'(x)?

Where h''' changes sign

	0		a		c		e	
h'''		$-$		$+$		$-$		$+$

$x = a, c, e$

D: [1, 3]

Example: Given $y = \sin^{-1}(x-2)$

- State intervals where y is concave up or concave down
- Find the inflection points

$$y' = \frac{1}{\sqrt{1-(x-2)^2}} \cdot (1) = \frac{1}{\sqrt{1-(x^2-4x+4)}} = \frac{1}{\sqrt{-x^2+4x-3}} = (-x^2+4x-3)^{-1/2}$$

$$y'' = -\frac{1}{2}(-x^2+4x-3)^{-3/2}(-2x+4) = \frac{-x+2}{(-x^2+4x-3)^{3/2}} = \frac{-x+2}{((-x+1)(x-3))^{3/2}}$$

$\underline{y''=0}$

$-x+2=0$

$x=2$

$\underline{y'' = \text{DNE}}$

$-x+1=0$

$x=1$ ← also endpts

$x-3=0$

$x=3$

	1		2		3
$(-x+2)$		$-$		$+$	
$((-x+1)(x-3))^{3/2}$		$+$		$+$	
y''		$-$		$+$	
y		CC↓		CC↑	

$\boxed{\begin{array}{l} \text{CC↓}: (1,2) \\ \text{CC↑}: (2,3) \\ \text{P.O.I.}: (2,0) \end{array}}$

$f(2) = \sin^{-1}(2-2)$
$= \sin^{-1} 0$
$= 0$

Example: Given $y = \frac{x^5}{5} - \frac{2x^3}{3} - 3x + 7$

- State intervals where y is concave up or concave down
- Find the inflection points

$y' = x^4 - 2x^2 - 3$

$y'' = 4x^3 - 4x = 4x(x^2-1)$

$\underline{y''=0}$

$4x(x+1)(x-1)=0$

$x = 0, 1, -1$

$\underline{y'' = \text{DNE}}$

none

		-1		0		1	
$4x$	$-$		$-$		$+$		$+$
$x+1$	$-$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$+$
y''	$-$		$+$		$-$		$+$
y	CC↓		CC↑		CC↓		CC↑

$\boxed{\begin{array}{l} \text{CC↑}: (-1,0)(1,\infty) \\ \text{CC↓}: (-\infty,-1),(0,1) \\ \text{P.O.I}: (-1, 157/15) \\ \quad (0,7) \\ \quad (1, 53/15) \end{array}}$

$f(-1) = 157/15$

$f(0) = 7$

$f(1) = 53/15$

4.3 part II: Second Derivative Test

Second Derivative Test for Local Extrema

max @ $x=c$, if $f'(c)=0$ and $f''(c)<0$ CCↆ

min @ $x=c$, if $f'(c)=0$ and $f''(c)>0$ CC↑

* can't use for f'=DNE

Example: Use 2nd Derivative Test to find extrema for $y = 4x^3 + 21x^2 + 36x - 20$.

$y' = 12x^2 + 42x + 36$

$= 6(2x+3)(x+2)$

$y' = 0$

$x = -3/2, -2$

$y'' = 24x + 42$

$y''(-3/2) = 6 > 0 \rightarrow$ min @ $x = -3/2$

$y''(-2) = -6 < 0 \rightarrow$ max @ $x = -2$

x	$f(x)$	
$-3/2$	-40.25	min b/c $f'=0$ and $f''>0$
-2	-40	max b/c $f'=0$ and $f''<0$

Example: Use the 2nd Derivative Test to find the extrema for $f(x) = xe^x$.

$f' = x \cdot e^x + e^x \cdot (1)$

$f' = e^x(x+1)$

$f' = 0$

$e^x = 0$
never

$x+1 = 0$

$x = -1$

$f'' = e^x(1) + (x+1)e^x$

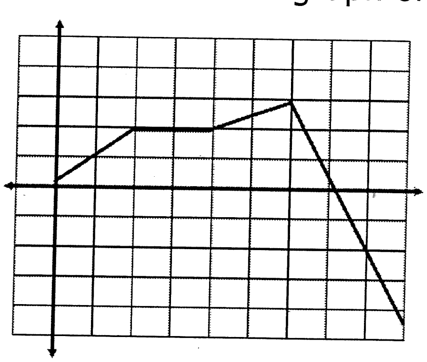
$f'' = e^x(1 + x + 1)$

$f'' = e^x(2+x)$

$f''(-1) = e^{-1}(2+-1) = \frac{1}{e} > 0 \rightarrow$ min @ $x=-1$

x	$f(x)$	
-1	$-1/e$	min b/c $f'=0$ and $f''>0$

Example: Below is a graph of a car's velocity.



(a) On what intervals is the car moving forward?

when $v>0$ $(0, 7\frac{1}{3})$

(b) When is the car's speed the greatest?

$t=9$

↳ |velocity|

(c) When is the car slowing down?

↳ when v and a have different signs

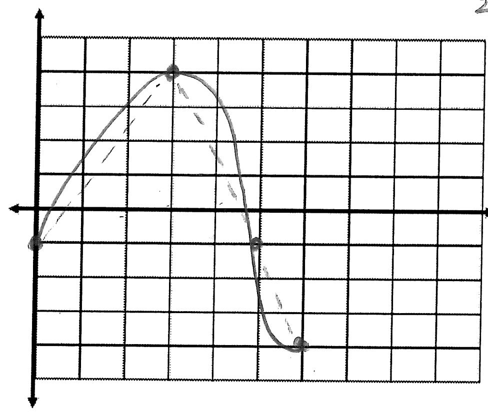
$(6, 7\frac{1}{3})$

ordered pairs

EXAMPLE: Suppose f(x) is continuous and differentiable on [0, 6] and satisfies:

1) Start w/ ordered pairs
2) connect points
3) change concavity

x	0	3	5	6
f	-1	4	-1	-3
f'	5	0	-8	0
f"	-1	-3	DNE	3



x = 3   x=5

x	0 < x < 3	3 < x < 5	5 < x < 6
f'	+ max	-	-
f"	-	- P.O.I	+
f	inc and CC↓	dec and CC↓	dec and CC↑

a) Identify any extrema on the interval [0,6] and classify them as maximums or minimums. Justify your answers.

max = 4 b/c f′ + to −
min = 0 b/c L.end and f′ > 0
min = 6 b/c R.end and f′ < 0

b) Identify any inflection points on the interval [0,6]. Justify your answers.

P.O.I = (5, −1) b/c f″ change sign

c) Sketch a possible graph of the function.

See above

EXAMPLE: Suppose f(x) is continuous on [-2, 4]. f(-2)=5, f(4)=1, and f′ and f″ have the following properties.

x	-2 < x < 0	x = 0	0 < x < 2	x = 2	2 < x < 4
f'	+	DNE	-	0	-
f"	+	DNE	+	0	-
f	inc & CC↑	max & cusp	dec & CC↑	P.O.I.	dec. & CC↓

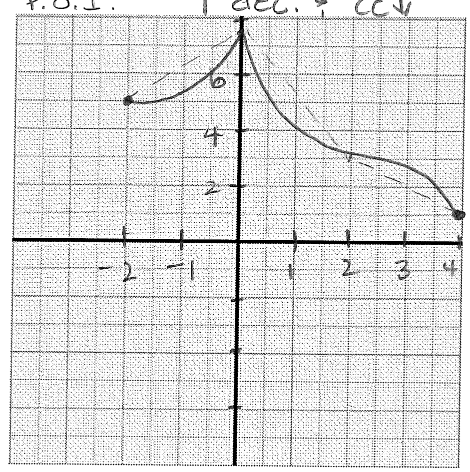
a) Find where all <u>absolute</u> extrema of f occur.

abs. min @ x = 4
abs. max @ x = 0

b) Find where the points of inflection of f occur.

P.O.I. @ x = 2

c) Sketch a possible graph of f.



4.4: Modeling and Optimization

Example: Find two positive numbers such that their product is 192 and their sum is a minimum.

Find?

min. sum of 2 pos. #s

$x + y = S$

$\frac{192}{y} + y = S$

Know?

$x \cdot y = 192$

$x = \frac{192}{y}$

Critical Values?

$S = \frac{192 + y^2}{y}$

$S' = \frac{y(2y) - (192 + y^2)(1)}{y^2}$

$S' = \frac{2y^2 - 192 - y^2}{y^2}$

$S' = \frac{y^2 - 192}{y^2}$

$S' = 0$

$y^2 - 192 = 0$

$y = \pm\sqrt{192}$ neg not in domain

$S' = DNE$

$y^2 = 0$

$y = 0$ not in domain

endpts

none

Chart

		$\sqrt{192}$	
$y^2 - 192$	−		+
y^2	+		+
S'	−		+
S	↘		↗

min @ $\boxed{y = \sqrt{192}}$

$x = \frac{192}{\sqrt{192}}$

$\boxed{x = \sqrt{192}}$

Strategy for "Optimizing" problems:

1. FIND? what are we finding max or min of? write equation.
2. KNOW? further info provided. draw a diagram.
3. CRITICAL VALUES? find when $f' = 0$, $f' = DNE$, and endpts
4. CHART make a sign chart to confirm your answer

Example: A manufacturer wants to design an open box with a square base and surface area of 108 square inches. What dimensions will produce a box with maximum volume?

Find?

dimensions w/ max volume of box

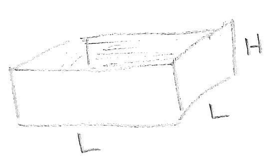
$V = L \cdot W \cdot H$

$V = L \cdot L \cdot H \quad V = L \cdot L \cdot \frac{108 - L^2}{4L} = \frac{1}{4}(108L - L^3)$

Know?

$SA = 108 \text{ in}^2 \rightarrow L^2 + 4HL = 108$

$H = \frac{108 - L^2}{4L}$



Critical Values?

$V' = \frac{1}{4}(108 - 3L^2)$

$V' = 0$

$\frac{1}{4}(108 - 3L^2) = 0$

$L^2 = \frac{108}{3}$

$L = \pm\sqrt{36}$, $L > 0$

$L = 6$

$V' = DNE$

none

endpts

none

Chart

		6	
$V' = \frac{1}{4}(108 - 3L^2)$	+		−
V	↗		↘

max @ $\boxed{L = 6''}$

$H = \frac{108 - 36}{24}$ $\boxed{H = 3''}$

EXAMPLE: A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square feet. What dimensions require the least amount of fencing if the river side does not require fence?

$\boxed{\text{Find?}}$

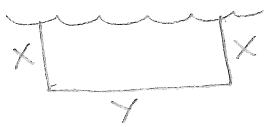
Min. perimeter of fence

$P = 2x + y$

$P = 2x + \frac{180{,}000}{x}$

$P = \frac{2x^2 + 180{,}000}{x}$

$\boxed{\text{Know?}}$



$A = 180{,}000$

$xy = 180{,}000$

$y = \frac{180{,}000}{x}$

$\boxed{\text{Critical Values?}}$

$P' = \frac{x(4x) - (2x^2 + 180{,}000)(1)}{x^2}$

$P' = \frac{4x^2 - 2x^2 - 180{,}000}{x^2} = \frac{2x^2 - 180000}{x^2}$

$P' = 0$

$2x^2 - 180{,}000 = 0$

$x = \pm\sqrt{90{,}000}$

$x = 300\text{ ft}$

$P' = DNE$

$x^2 = 0$

$x = 0$

not in domain

endpt

none

$\boxed{\text{Chart}}$

	300	
$2x^2 - 180{,}000$	$-$	$+$
x^2	$+$	$+$
P'	$-$	$+$
P	$\searrow$	$\nearrow$

min @ $\boxed{x = 300\text{ft}}$

$y = \frac{180{,}000}{300}$

$\boxed{y = 600\text{ ft.}}$

EXAMPLE: Suppose $r(x) = 8\sqrt{x}$ represents revenue and $c(x) = 2x^2$ represents cost, with x measured in thousands of units. Is there a production level that satisfies profit? If so, what is it? *note: profit = revenue - cost

$\boxed{\text{Find?}}$

units to max profit

$P = r - c$

$P = 8\sqrt{x} - 2x^2$

$\boxed{\text{Know?}}$

$r = 8\sqrt{x}$

$c = 2x^2$

$\boxed{\text{Critical Values}}$

$P = 8 \cdot x^{1/2} - 2x^2$

$P' = 4x^{-1/2} - 4x$

$P' = \frac{4}{x^{1/2}} - 4x$

$P' = \frac{4 - 4x^{3/2}}{x^{1/2}}$

$P' = 0$

$4 - 4x^{3/2} = 0$

$x^{3/2} = 1$

$x = 1$

$P' = DNE$

$x^{1/2} = 0$

$x = 0$

endpt

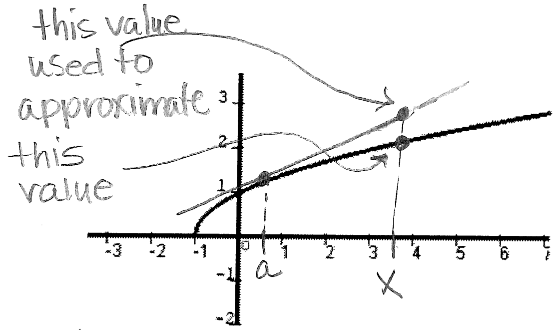
$x = 0$

$\boxed{\text{Chart}}$

	0	1	
$4 - 4x^{3/2}$		$+$	$-$
$x^{1/2}$		$+$	$+$
P'		$+$	$-$
P		$\nearrow$	$\searrow$

$\boxed{\text{max @ } x = 1 \text{ thousand units}}$

4.5: Linearization and Differentials



using tangent line to approximate value on a curve

eqn. of tangent line at pt. $x=a$

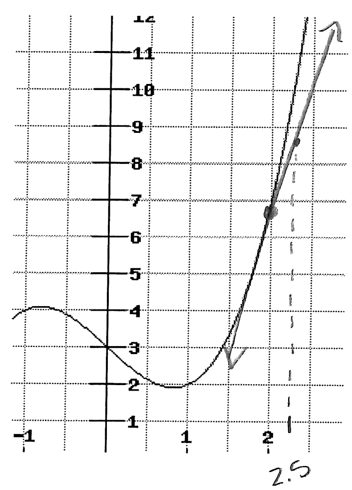
$$y - y_1 = m(x - x_1)$$
$$y - f(a) = f'(a)(x-a)$$
$$y = f(a) + f'(a)(x-a)$$

Definition: Linearization of f at a

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

no need to memorize this form... it is just equation of tan. line

EXAMPLE: Find $L(x)$ at $a = 2$ for $f(x) = x^3 - 2x + 3$.



$f(2) = 2^3 - 2(2) + 3 = 8 - 4 + 3 = 7 \longrightarrow (2,7)$

$f' = 3x^2 - 2$

$f'(2) = 3(4) - 2 = 12 - 2 = 10$

$y - 7 = 10(x-2)$

$\boxed{L(x) = 7 + 10(x-2)}$

- Find $L(2.1)$ and $f(2.1)$

$L(2.1) = 7 + 10(2.1 - 2)$
$= 7 + 10(.1) = \boxed{8}$ estimate

$f(2.1) = 2.1^3 - 2(2.1) + 3$
$= \boxed{8.061}$ actual

- Find $L(2.5)$ and f(2.5)

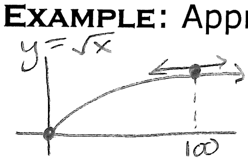
$L(2.5) = 7 + 10(2.5 - 2)$
$= 7 + 10(.5) = \boxed{12}$

$f(2.5) = 2.5^3 - 2(2.5) + 3$
$= \boxed{13.625}$

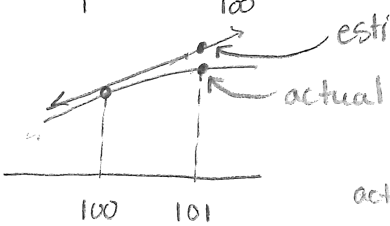
estimate gets less accurate further from $a = 2$

EXAMPLE: Approximate $\sqrt{101}$ using a linearization.

this is how your calculator works!



* use "known" pt. close to $x = 101$



actual: $\sqrt{101} \approx 10.04988$

$f(100) = \sqrt{100} = 10 \longrightarrow (100, 10)$

$f' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} \longrightarrow m = \frac{1}{20}$

$y - 10 = \frac{1}{20}(x - 100)$

$L(x) = 10 + \frac{1}{20}(x - 100)$

$L(101) = 10 + \frac{1}{20}(101 - 100)$
$= \boxed{10.05}$ est.

Definition: Differentials

Let y = f(*x*) be a differentiable function. The differential *dx* is an independent variable. The differential *dy* is

$$dy = f'(x)\,dx$$

like "Δx" change in x

like "Δy" change in y

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$dy = f'(x)dx$

like $\Delta y = m \cdot \Delta x$

$y - y_1 = m(x - x_1)$

Example: Find dy

- $y = x^4 + 23x$ $\quad \frac{dy}{dx} = 4x^3 + 23$

  $dy = (4x^3 + 23)dx$
- $y = \sin 3x$ $\quad dy = 3 \cdot \cos 3x \cdot dx$
- $y = 3x^2 + 5$ $\quad dy = 6x \cdot dx$

Example: Estimating change with differentials

The radius of a circle changes from $\underbrace{r = 6 \text{ to } r = 6.1 \text{ m}}$. Use dA to estimate the increase in the circle's area.

change in r = $dr = 0.1$

$A = \pi r^2$

$dA = 2\pi r\,dr$

$dA = 2\pi(6)(0.1) = \boxed{1.2\pi \text{ m}^2}$ estimates change in circle's area

Actual change

$A(6.1) - A(6) = \pi(6.1^2) - \pi(6^2) = 37.21\pi - 36\pi = \boxed{1.21\pi \text{ m}^2}$

- Find the amount of error.

Error $= |\text{actual} - \text{estimate}|$

$= |1.21\pi - 1.2\pi|$

$= \boxed{.01\pi}$

EXAMPLE: To estimate the depth of a well from the equation $s = 16t^2$ you time how long it takes a heavy stone to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

How much will your estimate be off if your time is off by 0.1 sec?

What if $t=5$? $s(5) = 16(5^2) = 400\text{ft.}$

$s(5.1) = 16(5.1)^2 = 416.16\text{ft}$

actual diff. @ $t=5$: 16.16ft.

close

$s = 16t^2$

$ds = 32t \cdot dt$

$ds = 32t(0.1)$

$\boxed{ds = 3.2t}$

estimate of error in depth (depends on t)

ds @ $t=5$
$= 3.2(5)$
$= 16$

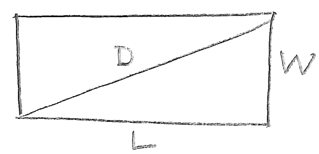
4.6: Related Rates

always rates w.r.t. time

EXAMPLE: The length of a rectangle is decreasing at the rate of 2 cm/sec while the width is increasing at a rate of 2 cm/sec.

When L = 12 and W = 5, find the rate of change of:

a) the area



$\frac{dL}{dt} = -2\text{cm/sec}$

$\frac{dW}{dt} = 2\text{cm/sec}$

$A = L \cdot W$

$$\frac{dA}{dt} = L \cdot \frac{dW}{dt} + W \cdot \frac{dL}{dt}$$

$$= 12(2) + 5(-2) = 24 - 10 = \boxed{14\text{ cm}^2/\text{sec}}$$

b) the perimeter

$P = 2L + 2W$

$$\frac{dP}{dt} = 2\frac{dL}{dt} + 2\frac{dW}{dt} = 2(-2) + 2(2) = -4 + 4 = \boxed{0\text{ cm/sec}}$$

c) the length of the diagonal

$D = \sqrt{W^2 + L^2} = (W^2 + L^2)^{1/2}$

$$\frac{dD}{dt} = \frac{1}{2}(W^2 + L^2)^{-1/2}\left(2W\frac{dW}{dt} + 2L\frac{dL}{dt}\right)$$

$$= \frac{2(5)(2) + 2(12)(-2)}{2\sqrt{5^2 + 12^2}} = \frac{20 + -48}{2(13)} = \frac{-28}{2(13)} = \boxed{\frac{-14}{13}\text{ cm/sec}}$$

Strategy for "Relate Rates" problems:

Find? +	When? +	Knowing? ⇒	Differentiate
• use math notation to name it • diagram	• list all info @ the instant in time	• write eqn. relating the variables	w.r.t. time

* amts that are changing cannot be subbed in until after differentiating

EXAMPLE: A 13 foot ladder is leaning against a house when its base starts to slide away. When the base is 12 ft from the house, the base is moving at a rate of 5ft/s.

a) How fast is the tip of the ladder sliding down the wall at that moment?

Find? + When? + Knowing? → Differentiate

$dy/dt = ?$

$x = 12$

$(y = 5)$

$dx/dt = 5\,ft/s$

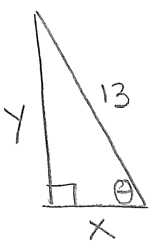
$x^2 + y^2 = 13^2$

$2x \cdot \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$2(12)(5) + 2(5)\,dy/dt = 0$

$120 + 10\,dy/dt$

$\boxed{dy/dt = -12\ ft/sec}$



b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing at that moment?

Find? When? Knowing? → Differentiate

$\frac{dA}{dt} = ?$

$x = 12$

$\frac{dx}{dt} = 5\,ft/s$

$A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}x \cdot \frac{dy}{dt} + y \cdot \frac{1}{2}\frac{dx}{dt}$

$\frac{dA}{dt} = \frac{1}{2}(12)(-12) + (5)(\frac{1}{2})(5)$

$\frac{dA}{dt} = -72 + 12.5 = \boxed{-59.5\ ft^2/sec}$

c) At what rate is the angle θ between the ladder and the ground changing at that moment?

Find? When? Knowing? → Differentiate

$\frac{d\theta}{dt} = ?$

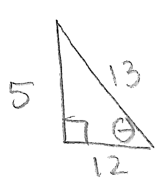
$x = 12$

$\frac{dx}{dt} = 5\,ft/sc$

$\cos\theta = \frac{x}{13}$

$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{13} \cdot \frac{dx}{dt}$

$-\frac{5}{13} \cdot \frac{d\theta}{dt} = \frac{1}{13} \cdot 5$



$\sin\theta = 5/13$

$\boxed{\frac{d\theta}{dt} = -1\ \text{radian/sec}}$

EXAMPLE: A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the equation s = $50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the base of launch pad. Find the rate of change in the angle of elevation for the camera at 10 sec. after lift-off.

Find? | When? | Knowing? → Differentiate

$\frac{d\theta}{dt} = ?$

$t = 10$ sec,
$s = 50 \cdot 10^2$
$= 5000$ ft

$s = 50t^2$
$\tan\theta = \frac{s}{2000}$

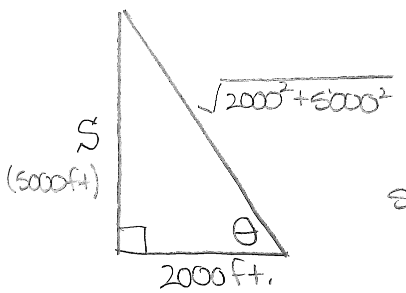
$s = 50t^2$
$\frac{ds}{dt} = 100t$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{2000} \cdot \frac{ds}{dt}$

$\left(\frac{\sqrt{2000^2+5000^2}}{2000}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{2000} \cdot 100(10)$

$\frac{d\theta}{dt} = \frac{1000}{2000} \cdot \left(\frac{2000^2}{2000^2+5000^2}\right)$

$\boxed{\frac{d\theta}{dt} = .069 \text{ rad/sec}}$



$\sec\theta = \frac{\sqrt{2000^2+5000^2}}{2000}$

EXAMPLE: Sand falls from a conveyor belt at the rate of 50 m^3/min onto the top of a conical pile. The height of the pile is always ⅜th the base diameter. How fast are the height and radius changing when the pile is 4 meters high?

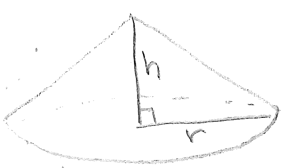
Find? | When? | Knowing? → Differentiate

$dh/dt = ?$
$dr/dt = ?$

$h = 4$

$4 = \frac{3}{4}r$
$\frac{16}{3} = r$

$\frac{dV}{dt} = 50 \text{ m}^3/\text{min}$
$h = \frac{3}{8}d$
$h = \frac{3}{8}(2r)$
$h = \frac{3}{4}r$
$V = \frac{1}{3}\pi r^2 h$
$V = \frac{\pi}{3}r^2\left(\frac{3}{4}r\right)$
$V = \frac{\pi}{4}r^3$



$\frac{dV}{dt} = \frac{\pi}{4} \cdot 3r^2 \cdot \frac{dr}{dt}$

$50 = \frac{3\pi}{4}\left(\frac{16}{3}\right)^2 \cdot \frac{dr}{dt}$

$50 = \frac{3\pi \cdot 256}{4 \cdot 3^2} \cdot \frac{dr}{dt}$

$50 = \frac{\pi \cdot 64}{3} \cdot \frac{dr}{dt}$

$\frac{50 \cdot 3}{64\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = .746 \text{ m/min}}$

$h = \frac{3}{4}r$

$\frac{dh}{dt} = \frac{3}{4} \cdot \frac{dr}{dt} = \frac{3}{4}\left(\frac{50 \cdot 3}{64\pi}\right)$

$\boxed{\frac{dh}{dt} = .5595 \text{ m/min}}$